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Sobrification and bicompletion of totally bounded quasi-uniform spaces

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1. *Abstract*. We observe that if $\mathscr{U}$ is a compatible totally bounded quasi-uniformity on a T_0-space $(X, \mathscr{T})$, then the bicompletion $(\tilde{X}, \tilde{\mathscr{U}})$ of $(X, \mathscr{U})$ is a strongly sober, locally quasicompact space. It follows that the b-closure S of $(X, \mathscr{U})$ in $(\tilde{X}, \tilde{\mathscr{U}})$ is homeomorphic to the sobrification of the space $(X, \mathscr{T})$. We prove that S is equal to $\tilde{X}$ if and only if $(X, \mathscr{T})$ is a core-compact space in which every ultrafilter has an irreducible convergence set and $\mathscr{U}$ is the coarsest quasi-uniformity compatible with $\mathscr{T}$. If $\mathscr{U}$ is the Pervin quasi-uniformity on X, then S is equal to $\tilde{X}$ if and only if X is hereditarily quasicompact, or equivalently, $\tilde{\mathscr{U}}$ is the Pervin quasi-uniformity on $\tilde{X}$.

We characterize the strongly sober locally quasicompact spaces as the topological T_0-spaces that admit a totally bounded bicomplete quasi-uniformity. Moreover, we note that every quasisober hereditarily quasicompact space admits a unique quasi-uniformity.

2. *Introduction*. We recall some basic results of the theory of quasi-uniform spaces. We refer the reader to [**5**] for the explanation of notions that are not explained here.

Let X be a non-empty set. A filter $\mathscr{U}$ on $X \times X$ is called a *quasi-uniformity* on X if (a) each member of $\mathscr{U}$ is a reflexive relation on X and (b) if $U \in \mathscr{U}$, then $V \circ V \subset U$ for some $V \in \mathscr{U}$. The pair $(X, \mathscr{U})$ is called a *quasi-uniform space*. If $U \in \mathscr{U}$ and $x \in X$, then $U(x)$ denotes the set $\{y \in X \mid (x, y) \in U\}$. The topology

$$\mathscr{T}(\mathscr{U}) = \{G \subset X \mid \text{for each } x \in G \text{ there exists } U \in \mathscr{U} \text{ such that } U(x) \subset G\}$$

is said to be *induced* by the quasi-uniformity $\mathscr{U}$ on X.

Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be quasi-uniform spaces. A map $f: X \to Y$ is called *quasi-uniformly continuous* if $(f \times f)^{-1}(V) \in \mathscr{U}$ whenever $V \in \mathscr{V}$. In this case $f:(X, \mathscr{T}(\mathscr{U})) \to (Y, \mathscr{T}(\mathscr{V}))$ is continuous ([**5**], proposition 1·14).

If a quasi-uniformity $\mathscr{U}$ on X has a base consisting of transitive relations, then $\mathscr{U}$ is called *transitive*. A quasi-uniformity $\mathscr{U}$ on X is called *totally bounded* if for each $U \in \mathscr{U}$ there is a finite cover $\mathscr{A}$ of X such that $\bigcup\{A \times A \mid A \in \mathscr{A}\} \subset U$. We will have to assume that the reader is familiar with the one-to-one correspondence between the totally bounded quasi-uniformities and the quasi-proximities that are compatible with a given topology on a set X ([**5**], 1·33). In particular, we recall that each quasi-uniformity $\mathscr{U}$ on a set X contains a totally bounded quasi-uniformity $\mathscr{V}$ such that $\mathscr{T}(\mathscr{U}) = \mathscr{T}(\mathscr{V})$.

If $\mathscr{U}$ is a quasi-uniformity on a set X, then $\mathscr{U}^{-1}$ denotes the quasi-uniformity $\{U^{-1} \mid U \in \mathscr{U}\}$ on X. Moreover $\mathscr{U}^*$ denotes the uniformity generated by $\{U \cap U^{-1} \mid U \in \mathscr{U}\}$. Note that $\mathscr{U}^*$ is the coarsest uniformity finer than both $\mathscr{U}$ and $\mathscr{U}^{-1}$.

Observe that a filter $\mathscr{F}$ on X is a $\mathscr{U}^*$-Cauchy filter if and only if for each $U \in \mathscr{U}$ there is an $F \in \mathscr{F}$ such that $F \times F \subset U$ ([**5**], proposition 3·2).

The quasi-uniformity $\mathscr{U}$ is called *bicomplete*, if $\mathscr{U}^*$ is complete. Note that $\mathscr{U}$ is totally bounded if and only if $\mathscr{U}^*$ is totally bounded ([**5**], 1·32).

In the following let $(X, \mathscr{U})$ be a quasi-uniform space such that $(X, \mathscr{T}(\mathscr{U}))$ is a T_0-space. The *bicompletion* $(\tilde{X}, \tilde{\mathscr{U}})$ of $(X, \mathscr{U})$ is constructed in ([**5**], 3·30). $(X, \mathscr{U})$ is quasi-uniformly embedded as a $\mathscr{T}(\tilde{\mathscr{U}}^*)$-dense subspace of $(\tilde{X}, \tilde{\mathscr{U}})$; we shall identify this subspace with $(X, \mathscr{U})$ ([**5**], theorem 3·33). Up to quasi-uniform isomorphism, $(\tilde{X}, \tilde{\mathscr{U}})$ is the only *bicompletion* of $(X, \mathscr{U})$, in the sense of being a bicomplete T_0-extension of $(X, \mathscr{U})$ in which $(X, \mathscr{U})$ is $\mathscr{T}(\tilde{\mathscr{U}}^*)$-dense ([**5**], 3·34). Moreover (though we shall not use this fact) the bicompletion is an epireflection in the category of T_0 quasi-uniform spaces (cf. [**25**], pp. 32, 68). Furthermore, we recall that $(\mathscr{U}^{-1})^{\sim} = (\tilde{\mathscr{U}})^{-1}$ and $(\mathscr{U}^*)^{\sim} = (\tilde{\mathscr{U}})^*$ ([**5**], 3·37) and that the topology $\mathscr{T}(\tilde{\mathscr{U}}^*)$ is compact if and only if $\mathscr{U}$ is totally bounded ([**5**], proposition 3·36).

If X is a topological space, the b-closure operator is the closure operator of the topology $\mathscr{T}(\mathscr{P}^*)$, where $\mathscr{P}$ denotes the Pervin quasi-uniformity on X (compare [**29**, **3**]). We recall that the Pervin quasi-uniformity for a topological space $(X, \mathscr{T})$ is generated by the subbase $\{G \times G \cup (X \backslash G) \times X \mid G \in \mathscr{T}\}$. Obviously it is transitive and totally bounded. It is known that the Pervin quasi-uniformity is the finest totally bounded quasi-uniformity that a topological space admits (see [**5**], p. 28).

We recall some notions and results of the theory of sober spaces. A *non-empty* subspace A of a topological space X is called *irreducible* ([**2**], chapter 2, §4) if each pair of non-empty A-open subsets has a non-empty intersection. A topological space X is called *quasisober*, if each closed irreducible subset is of the form $\overline{\{x\}}$ for some $x \in X$ ([**14**], p. 154). A quasisober T_0-space is called *sober*. A topological space is called *super-sober*, if the convergence set of every convergent ultrafilter is the closure of a unique point ([**7**], p. 310). It is well known (and obvious) that every super-sober T_1-space is a T_2-space. A topological space is called *strongly sober*, if it is super-sober and quasicompact ([**16**], p. 73). Let U and V be subsets of a topological space such that $U \subset V$. Then U is called *relatively quasicompact* ([**19**], p. 211) (bounded ([**21**], p. 326)) in V if every open cover of V has a finite subcollection that covers U. Note that U is relatively quasicompact in V if and only if every ultrafilter on V that contains U has a limit point in V (see [**7**], I, proposition 3·20).

A topological space is called *core-compact* (= quasi-locally-compact [**30**]), if every open set is the union of open sets that are relatively quasicompact in it ([**18**], p. 297; compare [**19**], p. 212). It is known that a topological space X is core-compact if and only if the lattice of its open sets is a continuous lattice ([**18**], proposition 4·2).

We shall use the usual notation $\ll$ for the 'way-below' relation (see e.g. [**7**], p. 38). It is clear that for open sets R and G in a topological space X, one has $R \ll G$ in the lattice of open sets of X if and only if R is relatively quasicompact in G.

A topological space is called *locally quasicompact*, if every point has a neighbourhood base consisting of quasicompact sets (see e.g. [**18**] and [**19**]). Obviously, every locally quasicompact space is core-compact. The sobrification of a T_0-space X is locally quasicompact if and only if X is core-compact ([**18**], theorem 4·5).

The reader may consult [**10**, **12**, **14**] for additional information about sober spaces and [**5**, **8**, **15**] for additional information about (quasi)-proximity structures and compactifications.

3. *Strongly sober locally quasicompact spaces.* It is known that if $\mathscr{U}$ is a compatible totally bounded quasi-uniformity on a T_1-space X, then $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$ is a T_1-space if and only if $\mathscr{U}$ is a uniformity ([**6**], proposition 2·1). On the other hand we have the following result (compare [**27**], pp. 500–502).

THEOREM 1. *Let X be a T_0-space and let $\mathscr{U}$ be a compatible totally bounded quasi-uniformity on X. Then $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$ is strongly sober and locally quasicompact.*

In the proof of Theorem 1 we will need the following lemma.

LEMMA 1. *Let $(X, \mathscr{U})$ be a quasi-uniform space. If x is a $\mathscr{T}(\mathscr{U}^*)$-limit point of a filter $\mathscr{G}$ on X, then $\mathrm{cl}_{\mathscr{T}(\mathscr{U})}\{x\}$ is the set of the $\mathscr{T}(\mathscr{U})$-cluster points and of the $\mathscr{T}(\mathscr{U})$-limit points of $\mathscr{G}$.*

Proof. Denote the set of the $\mathscr{T}(\mathscr{U})$-cluster points of $\mathscr{G}$ by F. Let $U \in \mathscr{U}$. Since x is a $\mathscr{T}(\mathscr{U}^*)$-limit point of $\mathscr{G}$, $(U \cap U^{-1})(x) \in \mathscr{G}$. Hence $F \subset \mathrm{cl}_{\mathscr{T}(\mathscr{U})}(U \cap U^{-1})(x) \subset U^{-2}(x)$. Thus $F \subset \bigcap\{U^{-2}(x) \mid U \in \mathscr{U}\} = \mathrm{cl}_{\mathscr{T}(\mathscr{U})}\{x\}$ ([**5**], proposition 1·8). Furthermore, since $\mathscr{T}(\mathscr{U}) \subset \mathscr{T}(\mathscr{U}^*)$, x is a $\mathscr{T}(\mathscr{U})$-limit point of $\mathscr{G}$. Thus $\mathrm{cl}_{\mathscr{T}(\mathscr{U})}\{x\} = F$. Clearly, F is also the set of the $\mathscr{T}(\mathscr{U})$-limit points of $\mathscr{G}$.

Proof of Theorem 1. If $U \in \tilde{\mathscr{U}}$ and $x \in \tilde{X}$, then there is a $V \in \tilde{\mathscr{U}}$ such that $V^2(x) \subset U(x)$. Thus $\mathrm{cl}_{\mathscr{T}(\tilde{\mathscr{U}}^{-1})} V(x) \subset U(x)$ and $\mathrm{cl}_{\mathscr{T}(\tilde{\mathscr{U}}^{-1})} V(x)$ is $\mathscr{T}(\tilde{\mathscr{U}}^*)$-quasicompact, because $\mathscr{T}(\tilde{\mathscr{U}}^{-1}) \subset \mathscr{T}(\tilde{\mathscr{U}}^*)$ and $\mathscr{T}(\tilde{\mathscr{U}}^*)$ is quasicompact. Since $\mathscr{T}(\tilde{\mathscr{U}}) \subset \mathscr{T}(\tilde{\mathscr{U}}^*)$, $\mathrm{cl}_{\mathscr{T}(\tilde{\mathscr{U}}^{-1})} V(x)$ is $\mathscr{T}(\tilde{\mathscr{U}})$-quasicompact. Therefore x has a $\mathscr{T}(\tilde{\mathscr{U}})$-neighbourhood base that consists of $\mathscr{T}(\tilde{\mathscr{U}})$-quasicompact sets. We conclude that $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$ is locally quasicompact.

Let $\mathscr{G}$ be an ultrafilter on $\tilde{X}$. Since $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}^*))$ is quasicompact, $\mathscr{G}$ has a $\mathscr{T}(\tilde{\mathscr{U}}^*)$-limit point x in $\tilde{X}$. By Lemma 1, $\mathrm{cl}_{\mathscr{T}(\tilde{\mathscr{U}})}\{x\}$ is the $\mathscr{T}(\tilde{\mathscr{U}})$-convergence set of $\mathscr{G}$. Because $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$ is a T_0-space, x is unique. Hence $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$ is strongly sober.

COROLLARY. *If $(X, \mathscr{U})$ is a totally bounded quasi-uniform T_0-space, then the b-closure S of $(X, \mathscr{U})$ in its bicompletion is the sobrification of $(X, \mathscr{T}(\mathscr{U}))$.*

Proof. Since a b-closed subspace of a sober space is sober ([**10**], p. 189), S is sober. Since a sober space is the (universal) sobrification of every b-dense subspace via its embedding ([**11**], p. 319), S is the sobrification of X via its embedding.

Example 1. Let $\mathscr{T}$ be the cofinite topology on the set of the positive integers N. The Pervin quasi-uniformity of $(N, \mathscr{T})$ is totally bounded. Hence it cannot be bicomplete, because $(N, \mathscr{T})$ is not sober. However, the fine quasi-uniformity $\mathscr{U}$ on $(N, \mathscr{T})$ is bicomplete. Of course, the fine quasi-uniformity on $(N, \mathscr{T})$ is not totally bounded ([**5**], example 2·38). In fact, the uniformity $\mathscr{U}^*$ is discrete.

Remark 1. The special case of the above corollary where $\mathscr{U}$ is the Pervin quasi-uniformity $\mathscr{P}$ on the T_0-space X can be found in ([**26**], p. 560) and ([**4**], proposition 5·8), where it was obtained from the fact that $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{P}}), \mathscr{T}(\tilde{\mathscr{P}}^{-1}))$ is the bitopological Stone–Čech compactification ([**25**], p. 23) of the bitopological space $(X, \mathscr{T}(\mathscr{P}), \mathscr{T}(\mathscr{P}^{-1}))$ ([**4**], p. 25).

In the rest of this section we will discuss some further aspects of Theorem 1. To this end we shall consider *partially ordered spaces* ('pospaces') $(X, \mathscr{T}, \leqslant)$ where $(X, \mathscr{T})$ is a topological space and $\leqslant$ a $\mathscr{T} \times \mathscr{T}$-closed partial order on X ([**7**], p. 272). We need the following basic facts about quasi-uniformities on these spaces, referring to ([**5**], chapter 4) for fuller information.

The pospace $(X, \mathscr{T}, \leqslant)$ is said to be *determined* by a quasi-uniformity $\mathscr{U}$ on X if $\mathscr{T} = \mathscr{T}(\mathscr{U}^*)$ and $\leqslant = \bigcap \mathscr{U}$. Note that $\mathscr{T}(\mathscr{U})$ is necessarily a T_0-topology. Every quasi-uniform T_0-space $(X, \mathscr{V})$ determines the pospace $(X, \mathscr{T}(\mathscr{V}^*), \bigcap \mathscr{V})$.

A compact pospace is determined by a unique quasi-uniformity $\mathscr{U}$ ([**5**], theorem 4·21). In this case $\mathscr{T}(\mathscr{U})$ is the upper topology and $\mathscr{T}(\mathscr{U}^{-1})$ is the lower topology of the pospace $(X, \mathscr{T}(\mathscr{U}^*), \bigcap \mathscr{U})$ ([**5**], proposition 4·22). Clearly $\mathscr{U}$ is totally bounded and bicomplete. Hence $\mathscr{T}(\mathscr{U})$ and $\mathscr{T}(\mathscr{U}^{-1})$ are strongly sober and locally quasicompact by Theorem 1.

A pospace $(X, \mathscr{T}, \leqslant)$ is called *totally order disconnected* [**24**] if, given $x, y \in X$ with $x \nleqslant y$, there exist disjoint $\mathscr{T}$-clopen sets U and V such that U is increasing, V is decreasing, $x \in U$, and $y \in V$. It is known that a compact pospace $(X, \mathscr{T}, \leqslant)$ is totally order disconnected if and only if its upper topology (equivalently, its lower topology) has a $\mathscr{T}$-clopen base ([**24**], remark on p. 515). Let us recall that Priestley's representation theory of distributive lattices [**24**] is based on the notion of a totally order disconnected compact pospace.

PROPOSITION 1. *A compact pospace is determined by a transitive quasi-uniformity if and only if it is totally order disconnected.*

In the proof of the proposition we will use the following lemma. It is folklore and occurs in several equivalent forms (e.g. [**27**], p. 500).

LEMMA 2. *Suppose that $\mathscr{U}$ and $\mathscr{V}$ are quasi-uniformities on a set X such that $\mathscr{T}(\mathscr{U}) = \mathscr{T}(\mathscr{V})$, $\mathscr{T}(\mathscr{U}^*) \subset \mathscr{T}(\mathscr{V}^*)$ and $\mathscr{T}(\mathscr{V}^*)$ is compact. Then $\mathscr{U} = \mathscr{V}$.*

Proof. Since $\mathscr{T}(\mathscr{V}^*)$ is Hausdorff, $\mathscr{T}(\mathscr{V})$ is T_0 and so is $\mathscr{T}(\mathscr{U})$. Hence $\mathscr{T}(\mathscr{U}^*)$ is Hausdorff. Since compact implies minimal Hausdorff, $\mathscr{T}(\mathscr{U}^*) = \mathscr{T}(\mathscr{V}^*)$. Further for each $x \in X$, by ([**5**], proposition 1·8) we have $\bigcap\{U^{-1}(x) \mid U \in \mathscr{U}\} = \mathrm{cl}_{\mathscr{T}(\mathscr{U})}\{x\} = \mathrm{cl}_{\mathscr{T}(\mathscr{V})}\{x\} = \bigcap\{V^{-1}(x) \mid V \in \mathscr{V}\}$. Hence $\bigcap \mathscr{U} = \bigcap \mathscr{V}$. Since a compact pospace is determined by a unique quasi-uniformity, $\mathscr{U} = \mathscr{V}$.

Proof of Proposition 1. Note that if a compact pospace $(X, \mathscr{T}, \leqslant)$ is determined by a transitive quasi-uniformity $\mathscr{U}$, then its upper topology $\mathscr{T}(\mathscr{U})$ has a $\mathscr{T}(\mathscr{U}^*)$-clopen base, namely $\{T(x) \mid T \in \mathscr{U}, T \text{ is transitive}, x \in X\}$.

On the other hand, if a compact pospace X is determined by a quasi-uniformity $\mathscr{U}$ such that its upper topology $\mathscr{T}(\mathscr{U})$ has a $\mathscr{T}(\mathscr{U}^*)$-clopen base $\mathscr{B}$, then the transitive quasi-uniformity $\mathscr{V}$ on X that is generated by the subbase $\{(X \backslash G) \times X \cup X \times G \mid G \in \mathscr{B}\}$ has the properties that $\mathscr{T}(\mathscr{V}^*) \subset \mathscr{T}(\mathscr{U}^*)$ and $\mathscr{T}(\mathscr{V}) = \mathscr{T}(\mathscr{U})$. By Lemma 2, $\mathscr{U} = \mathscr{V}$. Hence $\mathscr{U}$ is transitive.

PROPOSITION 2. *A topological T_0-space is strongly sober and locally quasicompact if and only if it admits a totally bounded bicomplete quasi-uniformity. Moreover, a strongly sober locally quasicompact space X admits a unique totally bounded bicomplete quasi-uniformity. It is the coarsest compatible quasi-uniformity on X.*

In the proof of Proposition 2 we will need the following facts, which we formulate as lemmas, because we will also use them in the next section.

LEMMA 3. *Let A, $B \subset X$. If $A\bar{\delta}B$ for each compatible quasi-proximity δ on a topological space X, then $[(X\backslash A)\times X \cup X\times(X\backslash B)]\in\mathscr{U}$ for each compatible (totally bounded) quasi-uniformity $\mathscr{U}$ on X.*

Proof. See [5], theorem 1·33.

LEMMA 4. *If a set R is relatively quasicompact in an open set G of a topological space X and δ is a compatible quasi-proximity on X, then $R\bar{\delta}X\backslash G$ (compare [5], proposition 1·43).*

Proof. For each $x\in G$ there is an open neighbourhood R_x of x such that $R_x\bar{\delta}X\backslash G$. Since R is relatively quasicompact in G, there exists a finite subcollection $\mathscr{R}'$ of $\{R_x \mid x\in G\}$ that covers R. Since $\bigcup\mathscr{R}'\bar{\delta}X\backslash G$, we conclude that $R\bar{\delta}X\backslash G$.

LEMMA 5. *A core-compact space admits a coarsest quasi-uniformity.*

Proof. Let X be a core-compact space. Since in a continuous lattice the way-below relation satisfies the interpolation property ([7], I, theorem 1·18), the filter $\mathscr{U}$ on $X\times X$ generated by the sets $(X\backslash R)\times X\cup X\times G$ (where R and G are open subsets of X such that $R \ll G$) is a quasi-uniformity on X (compare [5], proof of theorem 1·33). Clearly $\mathscr{U}$ is compatible. By Lemmas 4 and 3, $\mathscr{U}$ is the coarsest compatible quasi-uniformity on X.

Proof of Proposition 2. If a topological T_0-space X admits a totally bounded bicomplete quasi-uniformity, then, by Theorem 1, X is strongly sober and locally quasicompact.

Conversely, let X be a strongly sober locally quasicompact space and let $\mathscr{U}$ be the coarsest compatible quasi-uniformity on X (Lemma 5). If G_1 and G_2 are open subsets of X such that $G_1 \ll G_2$, then there is a quasicompact subset K of X such that $G_1\subset K\subset G_2$, because X is locally quasicompact (e.g. [7], I, proposition 1·4). By Lemmas 4 and 3 we conclude that $\{(X\backslash K)\times X\cup X\times G \mid K$ is quasicompact, G is open, and $K\subset G\}$ generates $\mathscr{U}$. Clearly $\mathscr{U}$ is totally bounded. In order to show that $\mathscr{U}$ is bicomplete, we show that the so-called patch topology $\mathscr{T}(\mathscr{U}^*)$ is compact ([7], p. 313; [9]): Let $\mathscr{H}$ be an ultrafilter on X. Then there is an $x\in X$ such that $\mathrm{cl}_{\mathscr{T}(\mathscr{U})}\{x\}$ is the $\mathscr{T}(\mathscr{U})$-convergence set of $\mathscr{H}$, because X is strongly sober. Since $\mathscr{T}(\mathscr{U}^*) = \sup\{\mathscr{T}(\mathscr{U}), \mathscr{T}(\mathscr{U}^{-1})\}$, we see that $\mathscr{H}$ converges to x in $(X, \mathscr{T}(\mathscr{U}^*))$, if we can show that $\mathscr{H}$ converges to x in $(X, \mathscr{T}(\mathscr{U}^{-1}))$. Hence it remains to prove that if K is quasicompact, G is open, $K\subset G$, and $U = (X\backslash K)\times X\cup X\times G$, then $U^{-1}(x)\in\mathscr{H}$. If $x\in G$, this is obvious. If $x\notin G$, then $U^{-1}(x) = X\backslash K$ and $K\cap \mathrm{cl}_{\mathscr{T}(\mathscr{U})}\{x\} = \varnothing$. Since K is quasicompact, we conclude that $K\notin\mathscr{H}$. Hence $X\backslash K = U^{-1}(x)\in\mathscr{H}$. Therefore, $\mathscr{U}$ is bicomplete.

Let $\mathscr{V}$ be a compatible totally bounded bicomplete quasi-uniformity on X. Then $\mathscr{U}\subset\mathscr{V}$. We note that $\mathscr{T}(\mathscr{U}^*)$ and $\mathscr{T}(\mathscr{V}^*)$ are compact topologies on X such that $\mathscr{T}(\mathscr{U}^*)\subset\mathscr{T}(\mathscr{V}^*)$. Moreover, $\mathscr{T}(\mathscr{U}) = \mathscr{T}(\mathscr{V})$. Hence by Lemma 2, $\mathscr{U} = \mathscr{V}$. Thus $\mathscr{U}$ is the only totally bounded bicomplete quasi-uniformity admitted by X.

Remark 2. (*a*) We observe that using the way-below relation we can easily describe the coarsest compatible quasi-proximity δ on a strongly sober locally quasicompact space X. For A, $B\subset X$, we have $A\bar{\delta}X\backslash B$ if and only if there are open subsets R and G of X such that $A\subset R\ll G\subset B$. This assertion is an immediate consequence of Lemma 4 and of the following two facts: $X \ll X$ and the way-below relation in X

is multiplicative (see e.g. [**20**]) (i.e. if G, H and K are open subsets of X such that $G \ll H$ and $G \ll K$, then $G \ll H \cap K$).

(*b*) Recall that for core-compact spaces X and Y a continuous map $f: X \to Y$ is called *perfect* ([**16**], p. 101) if whenever G_1 and G_2 are open subsets of Y such that $G_1 \ll G_2$, then $f^{-1}G_1 \ll f^{-1}G_2$. Note that if f is perfect, then (by the description of the coarsest compatible quasi-uniformity on a core-compact space given in the proof of Lemma 5) f is quasi-uniformly continuous with respect to the coarsest compatible quasi-uniformities on X and Y.

The following example shows that the converse does not obtain even if X is quasicompact. Equip $Y = N \cup \{\omega, \omega+1, \omega+2\}$ with the topology $\mathscr{T} = \{\{1, \ldots, n\} \mid n \in N\} \cup \{\varnothing, Y, N, Y\backslash\{\omega+1\}, Y\backslash\{\omega\}, N \cup \{\omega+2\}\}$. Let X denote the subspace $Y\backslash\{\omega+2\}$ of Y and let i denote the embedding of X in Y. Note that X and Y are locally quasicompact. Since X admits only one compatible totally bounded quasi-uniformity ([**22**], example 2), the coarsest compatible quasi-uniformity on X coincides with the Pervin quasi-uniformity $\mathscr{P}(X)$ on X ([**5**], 1·37). Let $\mathscr{P}(Y)$ denote the Pervin quasi-uniformity on Y. Clearly $i: (X, \mathscr{P}(X)) \to (Y, \mathscr{P}(Y))$ is quasi-uniformly continuous (e.g. [**5**], proposition 2·17). Hence i is quasi-uniformly continuous with respect to the coarsest compatible quasi-uniformities on X and Y. Let G be the open subset $Y\backslash\{\omega+1, \omega\}$ of Y. Then $G \ll G$ and $i^{-1}(G) = N$. Clearly, N is not quasicompact. Hence i is not perfect.

On the other hand, (by the description of the coarsest compatible quasi-proximity on a strongly sober locally quasicompact space given above) it is obvious that if X and Y are strongly sober and locally quasicompact, then $f: X \to Y$ is perfect if and only if f is quasi-uniformly continuous (compare [**5**], corollary 1·55) with respect to the coarsest compatible quasi-uniformities on X and Y.

(*c*) The strongly sober locally quasicompact spaces have been termed *stably quasicompact* by Hofmann ([**20**], p. 286). These spaces with the (continuous) perfect (= proper [**20**]) maps form a category which Hoffmann ([**16**], theorem 6·4), Wyler ([**31**], p. 631) and Hofmann ([**20**], theorem 1·5) have shown to be isomorphic to the category of compact pospaces and continuous isotone mappings. The one-to-one correspondence between the two classes of spaces is a transparent consequence of Proposition 2 above. Starting with a stably quasicompact space $(X, \mathscr{T})$ and taking the unique compatible totally bounded bicomplete quasi-uniformity $\mathscr{U}$, we have $(X, \mathscr{T}(\mathscr{U}^*), \cap\mathscr{U})$ as the corresponding compact pospace; the inverse correspondence is equally obvious.

The above two isomorphic categories are realizations of the category of (Eilenberg–Moore) algebras of a monad in TOP, the category of topological spaces. The monad occurs in several guises. Brümmer [**4**] introduced it as the monad with functor part $X \mapsto (\tilde{X}, \mathscr{T}(\tilde{\mathscr{P}}))$ where $\mathscr{P}$ denotes the Pervin quasi-uniformity on $X \in$ TOP. Simmons [**28**] then described it equivalently as the prime open filter monad and obtained its category of algebras as, in effect, the stably quasicompact topological spaces and (continuous) perfect maps. The category of algebras was reobtained independently by Wyler [**31**]. Further discussion of the stably quasicompact spaces may be found in [**1**], pp. 4, 9; [**19**], theorem 4·8; [**17**], p. 89; [**16**], pp. 73–75, 111 and [**27**], pp. 500–502.

A topological space X stays fixed under the functor $X \mapsto (\tilde{X}, \mathscr{T}(\tilde{\mathscr{P}}))$ if and only if the b-topology of X is compact ([**4**], theorem 5·1), i.e. if and only if X is sober and hereditarily quasicompact ([**14**], theorem 3.1). We return to such spaces in section 5.

4. *The bicompletion as sobrification.* In the proof of the main result of this section (Theorem 2) we will need the following two lemmas.

LEMMA 6. *Let Y be a b-dense subspace of a topological space X and let δ be a compatible quasi-proximity on Y. Then there is a unique compatible quasi-proximity δ' on X such that $\delta' \mid Y = \delta$.*

Furthermore, δ is the coarsest compatible quasi-proximity on Y if and only if δ' is the coarsest compatible quasi-proximity on X. In particular, Y admits a coarsest quasi-proximity if and only if X admits a coarsest quasi-proximity.

Proof. For $A, B \subset X$ set $A\bar{\delta'}B$, if there exist an open set G of X and a closed set F of X such that $A \subset G$, $B \subset F$, $G \cap F = \varnothing$, and $G \cap Y\bar{\delta}F \cap Y$. We leave it to the reader to check that δ' is a quasi-proximity on X (see [**5**], p. 10). Let us show that δ' is compatible. Let $x \in X$ and $A \subset X$. If $x\bar{\delta'}A$, then, clearly, $x \notin \bar{A}$. On the other hand, assume that $x \notin \bar{A}$. Choose $y \in (\overline{\{x\}}\backslash \bar{A}) \cap Y$. Then there is a Y-open neighbourhood G of y such that $G\bar{\delta}\bar{A} \cap Y$. Let G' be the unique open set of X such that $G' \cap Y = G$. Since G' is a neighbourhood of y and $y \in \overline{\{x\}}$, we have that $x \in G'$. Clearly $G' \cap \bar{A} = \varnothing$. Using $G\bar{\delta}\bar{A} \cap Y$ we see that $x\bar{\delta'}A$. We conclude that δ' is compatible.

Next we want to show that δ' is the unique compatible quasi-proximity on X such that $\delta' \mid Y = \delta$. Assume that X admits two different quasi-proximities δ_1 and δ_2 extending δ. We can assume that there are an open set G of X and a closed set F of X such that $G\delta_1 F$, but $G\bar{\delta}_2 F$. Since $G\bar{\delta}_2 F$, we have that $Y \cap G\bar{\delta}_2\, Y \cap F$. Since δ_1 and δ_2 extend δ, we conclude that $Y \cap G\bar{\delta}\, Y \cap F$, and thus $Y \cap G\bar{\delta}_1\, Y \cap F$. Therefore, there exist an open set G' of X containing $Y \cap G$ and a closed set F' of X containing $Y \cap F$ such that $G'\bar{\delta}_1 F'$. Set $G'' = G \cap G'$ and $F'' = F \cap F'$. Then $G''\bar{\delta}_1 F''$, $Y \cap G'' = Y \cap G$, and $Y \cap F'' = Y \cap F$. Since Y is b-dense, we conclude that $G'' = G$ and $F'' = F$ – a contradiction.

Then δ' is the unique compatible quasi-proximity on X such that $\delta' \mid Y = \delta$. We conclude that if ρ is a compatible quasi-proximity on X, then $(\rho \mid Y)' = \rho$.

The remaining assertions follow from the results proved above and the fact that whenever δ_1 and δ_2 are compatible quasi-proximities on Y, then δ_1 is coarser than δ_2 if and only if δ_1' is coarser than δ_2'.

Observe that it will follow from Proposition 3(a) that the quasi-uniformity $\mathscr{U}$ described in Example 1 cannot be extended to the sobrification of $\mathscr{T}(\mathscr{U})$.

We call a topological space X *strongly sobrifiable* if every ultrafilter on X has an irreducible convergence set. The following result justifies this term.

LEMMA 7. *Let X be a T_0-space. Then X is strongly sobrifiable if and only if its sobrification S is strongly sober.*

Proof. Let X be strongly sobrifiable. Consider an ultrafilter $\mathscr{G}$ on S and let F be its convergence set in S. Let $\mathscr{B} = \{A \cap X \mid A \in \mathscr{G}, A \text{ is open or closed in } S\}$. Then $\mathscr{B}$ has the finite intersection property, because X is b-dense in S. Consider an ultrafilter $\mathscr{H}$ on X containing $\mathscr{B}$. Clearly, its convergence set in X is $F \cap X$. Since X is strongly sobrifiable, $F \cap X$ is irreducible in X. Since X is dense in S, F is irreducible in S and by the sobriety of S there is a unique $a \in S$ with $\overline{\{a\}} = F$. Thus S is strongly sober.

Conversely, let S be strongly sober. Let $\mathscr{G}$ be an ultrafilter on X and let C be its convergence set in X. The filter base $\mathscr{G}$ generates an ultrafilter $\mathscr{G}'$ on S. Since S is strongly sober, there is a unique $y \in S$ such that $\overline{\{y\}}$ is the convergence set of $\mathscr{G}'$ in S. Clearly $C = X \cap \overline{\{y\}}$. Since y is in the b-closure of X one sees by a standard argument (see e.g. [**11**], p. 319) that C is irreducible in X. Thus X is strongly sobrifiable.

Theorem 2. *Let X be a T_0-space.*

(*a*) *Let $\mathscr{U}$ be a compatible totally bounded quasi-uniformity on X. If $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$ is the sobrification of X, then X is core-compact and strongly sobrifiable and $\mathscr{U}$ is the coarsest compatible quasi-uniformity on X.*

(*b*) *Let X be core-compact and strongly sobrifiable. Let $\mathscr{U}$ denote the coarsest compatible quasi-uniformity on X (Lemma 5). Then $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$ is the sobrification of X.*

Proof. (*a*) Since the sobrification of X is locally quasicompact and strongly sober (Theorem 1), X is core-compact ([**18**], theorem 4·5) and strongly sobrifiable (Lemma 7). Since $\tilde{\mathscr{U}}$ is the coarsest compatible quasi-uniformity on $\tilde{X}$ (Proposition 2), we have that $\tilde{\mathscr{U}} \mid X = \mathscr{U}$ is the coarsest compatible quasi-uniformity on X by Lemma 6.

(*b*) Let S be the b-closure of X in $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$. By the Corollary to Theorem 1, S is the sobrification of X. Since X is core-compact, S is locally quasicompact ([**18**], theorem 4·5). S is strongly sober by Lemma 7. By Lemma 6 $\tilde{\mathscr{U}} \mid S$ is the coarsest compatible quasi-uniformity on S. By Proposition 2 $\tilde{\mathscr{U}} \mid S$ is bicomplete. Hence $\tilde{X} = S$ and $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$ is the sobrification of X.

Corollary. *For a T_0-space X the following conditions are equivalent.*

(*a*) *X is core-compact and strongly sobrifiable.*

(*b*) *The sobrification of X is locally quasicompact and strongly sober.*

(*c*) *X has a coarsest compatible quasi-uniformity $\mathscr{U}$ and $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$ is the sobrification of X.*

5. *Hereditarily quasicompact sober spaces.* The main result of this section will answer the question: When is the bicompletion of a Pervin structure again a Pervin structure?

We begin this section with an observation about hereditarily quasicompact quasisober spaces.

Proposition 3. (*a*) *Every hereditarily quasicompact quasisober space admits a unique quasi-uniformity, namely the Pervin quasi-uniformity, and this quasi-uniformity is bicomplete.*

(*b*) *Every hereditarily quasicompact space is strongly sobrifiable.*

Proof. (*a*) Let X be a hereditarily quasicompact quasisober space and let $\mathscr{U}$ be a compatible quasi-uniformity on X. Recall that there is a unique compatible totally bounded quasi-uniformity on a hereditarily quasicompact space, namely the Pervin quasi-uniformity ([**5**], theorem 2·36). Hence it suffices to show that $\mathscr{U}$ is totally bounded. Recall also that a quasi-uniform space $(Y, \mathscr{V})$ is totally bounded if and only if each ultrafilter on Y is a $\mathscr{V}^*$-Cauchy filter ([**5**], proposition 3·14).

Let $\mathscr{G}$ be an ultrafilter on X. We show that $\mathscr{G}$ is a $\mathscr{U}^*$-Cauchy filter. Set $F_0 = \bigcap\{F \in \mathscr{G} \mid F \text{ is closed in } X\}$. Since X is hereditarily quasicompact and $\mathscr{G}$ is closed under finite intersections, $F_0 \in \mathscr{G}$. Clearly, if G is open in X and $G \cap F_0 \neq \varnothing$, then

$G \cap F_0 \in \mathscr{G}$. Hence F_0 is irreducible. Since X is quasisober, we have that $F_0 = \mathrm{cl}_{\mathscr{T}(\mathscr{U})}\{x\}$ for some $x \in X$. Let $U \in \mathscr{U}$. Note that $\mathrm{cl}_{\mathscr{T}(\mathscr{U})}\{x\} \subset U^{-1}(x)$ and that $x \in F_0 \cap \mathrm{int}_{\mathscr{T}(\mathscr{U})} U(x)$. Hence $U(x) \cap U^{-1}(x) = (U \cap U^{-1})(x) \in \mathscr{G}$. We conclude that x is a $\mathscr{T}(\mathscr{U}^*)$-limit point of $\mathscr{G}$. Hence $\mathscr{G}$ is a $\mathscr{U}^*$-Cauchy filter on X. Thus $\mathscr{U}$ is totally bounded. Therefore, $\mathscr{U}$ is the Pervin quasi-uniformity for X. In fact we have shown that the Pervin quasi-uniformity of X is bicomplete.

(*b*) The assertion follows from the proof given above.

Remark. 3. Observe that a topological space with a finite topology is hereditarily quasicompact and quasisober. An uncountable set equipped with the cofinite topology is an example of a topological space that admits a unique quasi-uniformity ([**5**], example 2·37), but is not sober. As we mentioned above, the cofinite topology on N admits a quasi-uniformity that is not totally bounded ([**5**], example 2·38), although it is hereditarily quasicompact.

Recently, the first author could prove the following two results. A topological space admits a unique quasi-uniformity if and only if it is a hereditarily quasicompact space that has no strictly decreasing sequence (of order type ω) of open sets with open intersection ([**23**], see the remark following the theorem). A topological space admits a unique quasi-proximity if and only if its topology is the unique base of open sets that is closed under finite unions and finite intersections [**22**].

THEOREM 3. *Let $\mathscr{U}$ be a compatible totally bounded quasi-uniformity on a topological T_0-space X. Denote the Pervin quasi-uniformity of $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$ by $\mathscr{Q}$. Then the following conditions are equivalent*:

(*a*) *X is hereditarily quasicompact.*

(*b*) *$\mathscr{U}$ is the Pervin quasi-uniformity and X is b-dense in $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$.*

(*c*) *$\mathscr{U}$ is the Pervin quasi-uniformity and $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$ is the sobrification of X.*

(*d*) *$\tilde{\mathscr{U}} = \mathscr{Q}$.*

(*e*) *$\mathscr{T}(\tilde{\mathscr{U}}^*) = \mathscr{T}(\mathscr{Q}^*)$.*

Proof. $(a) \to (b)$: As we have mentioned above, if X is hereditarily quasicompact, then the Pervin quasi-uniformity is the only compatible totally bounded quasi-uniformity on X ([**5**], theorem 2·36). Hence the Pervin quasi-uniformity $\mathscr{U}$ is the coarsest compatible quasi-uniformity on X ([**5**], 1·37). Clearly, a hereditarily quasicompact space is locally quasicompact. By Proposition 3 (*b*) a hereditarily quasicompact space is strongly sobrifiable. Therefore the assertion follows from Theorem 2.

$(b) \to (c)$: This implication is an immediate consequence of Theorem 1.

$(c) \to (d)$: It is known that if $\mathscr{P}$ is the Pervin quasi-uniformity for a topological space Z and Y is a subspace of Z, then $\mathscr{P} \mid Y$ is the Pervin quasi-uniformity for Y ([**5**], 3·20). Hence the result follows from Lemma 6, because $\tilde{\mathscr{U}} \mid X = \mathscr{U}$ and X is b-dense in $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$.

$(d) \to (e)$: This is obvious.

$(e) \to (a)$: By ([**14**], theorem 3·1) $\mathscr{T}(\mathscr{Q})$ is hereditarily quasicompact and sober, because $\mathscr{T}(\mathscr{Q}^*)$ is compact. Since $\mathscr{Q}$ is the Pervin quasi-uniformity for $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{U}}))$, $\mathscr{T}(\tilde{\mathscr{U}}) = \mathscr{T}(\mathscr{Q})$. Hence the subspace X of $\tilde{X}$ is hereditarily quasicompact.

Recall that hereditarily quasicompact sober spaces are sometimes called *Zariski spaces*. They have the property that each closed subset is a finite union of point closures ([**13**], p. 411).

Example 2. Note that for every ordinal α the (compact) order topology on $\alpha+1$ is determined by the Pervin quasi-uniformity $\mathscr{P}$ of the topology $\{(\beta,\alpha] \mid \beta \in \alpha\} \cup \{\varnothing, \alpha+1\}$ on $\alpha+1$. Let $\alpha \geqslant \omega$. Then $(\alpha+1, \mathscr{T}(\mathscr{P}^{-1}))$ is strongly sober and locally quasicompact (by Theorem 1), but $\mathscr{P}^{-1}$ is not the Pervin quasi-uniformity of $\mathscr{T}(\mathscr{P}^{-1})$ (by Theorem 3), because $\mathscr{T}(\mathscr{P}^{-1})$ is not hereditarily quasicompact. Since $\mathscr{P}^{-1}$ is totally bounded and bicomplete, $\mathscr{P}^{-1}$ is the coarsest $\mathscr{T}(\mathscr{P}^{-1})$-compatible quasi-uniformity (Proposition 2).

Proposition 4. *A topological space X is hereditarily quasicompact if and only if it is strongly sobrifiable and admits a unique quasi-proximity.*

Proof. Let X be a strongly sobrifiable space that admits a unique quasi-proximity. Since X is strongly sobrifiable and since irreducible sets are non-empty by definition, X is quasicompact. Let G be a proper non-empty open subset of X and let $\mathscr{G}$ be an ultrafilter on X that contains G. Since X admits a unique quasi-proximity, there is a non-empty finite collection $\mathscr{M}$ of open subsets of X such that $\bigcap \mathscr{M} \subset G$ and such that each element of $\mathscr{M}$ contains a limit point of $\mathscr{G}$ ([**22**], proposition). Since the convergence set of $\mathscr{G}$ is irreducible, we see that $\mathscr{G}$ has a limit point in G. Hence each open subspace of X is quasicompact. We conclude that X is hereditarily quasicompact.

The converse follows from Proposition 3(*b*) and ([**5**], theorem 2·36).

In light of Proposition 3(*a*) we have the following corollary.

Corollary. *If a strongly sober space admits a unique quasi-proximity, then it admits a unique quasi-uniformity.*

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Added in proof. Recently the first author proved the following result. A topological space that admits a coarsest quasi-uniformity and in which each convergent ultrafilter has an irreducible convergence set is core-compact (Corollary 3, Topological spaces with a coarsest compatible quasi-proximity, Quaestiones Math. (to appear)).